International Journal of Theoretical Physics, Vol. 4, No. 1 (1971), pp. 1–9

Inertial Reference Frames in Einstein's Theory of Gravitation

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Received: 12 *October* 1970

Abstract

A physical definition of the inertial reference frame (IRF) is given, and the properties of solutions of the Einstein equation (with cosmological constant Λ), which admit an IRF (IRF solutions) are investigated. Their Petrov type is uniquely determined by the viscous stress tensor. Only the types *I*, *D* or 0 are possible. The unique vacuum IRF solution is the Minkowski space-time. The unique IRF solution belonging to a perfect fluid is the Einstein universe. Λ is of special importance. For $\Lambda = 0$, the only physically admissible IRF solution is the Minkowski space-time. For $\Lambda \neq 0$, only interior solutions with strong restrictions for density and pressure are possible.

1. *Introduction*

A concept of central meaning in *Newtonian physics* is the inertial reference frame (IRF). It may be defined as such a frame, in which no inertial forces occur. It is characterised, moreover, by the fact that in referring to it, all physical laws receive their most simple mathematical form.

On the other hand, in *Einstein's theory* of curved space-time (including the flat world as a limiting case), gravitational and inertial forces are no longer primary concepts. On the contrary, the starting point of the theory is the replacement of both by geometrical ideas (for example, the geodetic movement of test particles). Furthermore, physical laws are formulated independently of reference frames.

Nevertheless, general relativity too needs the concept of the reference frame, because the results of measurement always depend on the motion of the respective observer. Accordingly, reference frames—represented by observer fields—are necessary to establish a connection between the mathematical quantities (defined independently of a reference frame) and the measured variables (dependent on the observer). Consequently, as elements of a generally covariant *theory of measurement* (compare, for example, Uhlmann, 1960; Dehnen, 1970), reference frames form an essential part of general relativity.

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Although the formulation of the theory does not mark out any reference frame, comparison with Newtonian physics raises the question of definition, existence and remaining role of inertial reference frames in Einstein's theory.

It appears that only special solutions of Einstein's field equations admit an IRF (IRF solutions). In the following we discuss the physical and mathematical properties of these IRF solutions and of the corresponding distributions of matter. Hereby the cosmological constant Λ will be of special importance; and in this way one aspect of the physical meaning of Λ will become more transparent.

2. *IRF and General Properties of IRF Solutions*

2.1. *Definition*

Following Newtonian mechanics, we define an IRF in a physical way using a cloud of test particles; *a cloud of free test particles which moves rigidly and without rotation represents an inertial reference frame.* That means, that—in a region with a non-vanishing 4-volume—the 4-velocity field $u^\alpha(x)$ of the particles ($u^\alpha u_\alpha = +1$) has the following kinematical properties:†

$$\dot{u}^\alpha := u^\alpha_{;\epsilon} u^\epsilon = 0 \qquad \text{(vanishing acceleration)} \tag{2.1.1}$$

$$\theta := u^\epsilon_{;\epsilon} = 0 \qquad \text{(vanishing expansion)} \tag{2.1.2}$$

$$\sigma_{\alpha\beta} := u_{(\alpha;\beta)} - \dot{u}_{(\alpha} u_{\beta)} - \tfrac{1}{3}(g_{\alpha\beta} - u_\alpha u_\beta) . \theta = 0 \qquad \text{(vanishing shear)} \tag{2.1.3}$$

$$\omega_{\alpha\beta} := u_{[\alpha;\beta]} - \dot{u}_{[\alpha} u_{\beta]} = 0 \qquad \text{(vanishing rotation)} \tag{2.1.4}$$

According to the identity

$$u_{\alpha;\beta} \equiv \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \tfrac{1}{3}(g_{\alpha\beta} - u_\alpha u_\beta)\theta + \dot{u}_\alpha u_\beta \tag{2.1.5}$$

the 4-velocity field $u^\alpha(x)$ is covariantly constant in the whole region

$$u_{\alpha;\beta} = 0 \tag{2.1.6}$$

We obtain a *dynamic interpretation* of an IRF in considering the u^α-congruence as the world-lines of observers. 3-momentum and energy of any freely moving test particle, as measured by these inertial observers, do not change with time, i.e. the particle moves rectilinearly with constant velocity. This fact can be interpreted as the absence of gravitational and any inertial forces with respect to an IRF (Dehnen, 1970).

2.2. *Relation Between $T_{\alpha\beta}$ and $R_{\alpha\sigma\beta\nu}$*

We call IRF solutions those solutions of Einstein's field equations (Λ is the cosmological constant)

$$R_{\alpha\beta} - \tfrac{1}{2}R g_{\alpha\beta} - \Lambda g_{\alpha\beta} = -T_{\alpha\beta} \tag{2.2.1}$$

† Signature of $g_{\alpha\beta}$: ($---+$). Range of indices: $\alpha, \beta, \ldots = 1, 2, 3, 4$. A definition is indicated by $:=$ and an identity by $\equiv$. Partial and covariant derivatives are denoted by subscripts, and ; respectively.

$$a_{(\alpha\beta)} := \tfrac{1}{2}(a_{\alpha\beta} + a_{\beta\alpha}), \qquad a_{[\alpha\beta]} := \tfrac{1}{2}(a_{\alpha\beta} - a_{\beta\alpha})$$

which admit an IRF. In consequence of the Ricci identity and of (2.1.6), being valid in a 4-region, the metric of an IRF solution satisfies

$$R_{\alpha\sigma\beta\nu} u^{\alpha} = 0 \tag{2.2.2}$$

and therefore

$$R_{\alpha\beta} u^{\alpha} = 0 \tag{2.2.3}$$

Contracting the decomposition of the Riemann tensor

$$R_{\alpha\sigma\beta\nu} \equiv C_{\alpha\sigma\beta\nu} + g_{\alpha[\beta} R_{\nu]\sigma} + g_{\sigma[\nu} R_{\beta]\alpha} + \tfrac{1}{3} g_{\alpha[\nu} g_{\beta]\sigma} R \tag{2.2.4}$$

with $u^{\nu} u^{\sigma}$, and taking into account (2.2.2) and (2.2.3), we can write (2.2.4) in view of (2.2.1) in the form

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} - \Lambda g_{\alpha\beta} = -\frac{R}{3} u_{\alpha} u_{\beta} - \left(\frac{R}{6} + \Lambda\right) g_{\alpha\beta} - 2 C_{\alpha\sigma\beta\nu} u^{\sigma} u^{\nu} \tag{2.2.5}$$

Subtracting (2.2.1) from (2.2.5), we finally obtain for IRF-solutions the following *fundamental relation between the components of the energy momentum tensor* (*compare, for example, Eckart,* 1940; *Audretsch,* 1967) *and the Riemann tensor*:

$$\begin{aligned} T_{\alpha\beta} &= (\rho + p) u_{\alpha} u_{\beta} - p g_{\alpha\beta} + q_{\alpha} u_{\beta} + q_{\beta} u_{\alpha} + \theta_{\alpha\beta} \\ &= \frac{R}{3} u_{\alpha} u_{\beta} + \left(\frac{R}{6} + \Lambda\right) g_{\alpha\beta} - 2 E_{\alpha\beta} \end{aligned} \tag{2.2.6}$$

with energy density ρ, hydrostatic pressure p, heat flow q^{α}

$$q_{\alpha} := T^{\kappa\lambda} u_{\kappa} (g_{\alpha\lambda} - u_{\alpha} u_{\lambda}) \tag{2.2.7}$$

and viscous stress tensor $\theta_{\alpha\beta}$, all as measured by the observer field $u^{\alpha}(x)$. $E_{\alpha\beta}$ defined by

$$E_{\alpha\beta} := -C_{\alpha\sigma\beta\nu} u^{\sigma} u^{\nu} \tag{2.2.8}$$

has the same properties

$$E^{[\alpha\beta]} = 0, \qquad E^{\alpha\beta} u_{\beta} = 0, \qquad E_{\epsilon}{}^{\epsilon} = 0 \tag{2.2.9a–c}$$

as $\theta_{\alpha\beta}$:

$$\theta^{[\alpha\beta]} = 0, \qquad \theta^{\alpha\beta} u_{\beta} = 0, \qquad \theta_{\epsilon}{}^{\epsilon} = 0 \tag{2.2.10a–c}$$

Because of the vanishing divergence of $T^{\alpha\beta}$, we get from (2.2.6) according to (2.1.6) and (2.2.9b) the differential equation

$$R_{,\epsilon} u^{\epsilon} = 0 \tag{2.2.11a}$$

Furthermore, we obtain from (2.2.6) by contracting with u^{β} and using (2.2.9b) and (2.2.10b) the following relations:

$$q^{\alpha} = 0 \tag{2.2.11b}$$

$$\rho - \tfrac{1}{2} R - \Lambda = 0 \tag{2.2.11c}$$

and by contracting with $g^{\alpha\beta} - u^\alpha u^\beta$ and using (2.2.9c) and (2.2.10c)

$$p + \frac{R}{6} + \Lambda = 0 \tag{2.2.11d}$$

$$\theta_{\alpha\beta} + 2E_{\alpha\beta} = 0 \tag{2.2.11e}$$

The necessary conditions (2.2.11a–e) *characterise the IRF-solutions.* The consequences following from (2.2.11a–e) will be discussed below.

2.3. *Petrov Type*

The connection (2.2.11e) relates the viscous stress tensor $\theta^{\alpha\beta}$ to the first five linearly independent components of the Weyl tensor, which are represented by $E^{\alpha\beta}$. The remaining five components are given by†

$$H_{\alpha\beta} := -{}^*C_{\alpha\sigma\beta\nu}u^\sigma u^\nu; \qquad H_{[\alpha\beta]} = H_{\alpha\beta}u^\alpha = H_\epsilon{}^\epsilon = 0 \tag{2.3.1}$$

From (2.2.2), (2.2.3) and (2.2.4) follows

$$u_{[\lambda}C_{\alpha\sigma]\beta\nu}u^\nu = 0 \tag{2.3.2}$$

and therefore

$$H_{\alpha\beta} = -{}^*C_{\alpha\sigma\beta\nu}u^\sigma u^\nu = 0 \tag{2.3.3}$$

In classifying the Weyl tensor according to Petrov's matrix method (Petrov, 1954; compare Jordan *et al.*, 1960, and Anderson, 1967), one projects

$$C_{\alpha\beta} := E_{\alpha\beta} + iH_{\alpha\beta} \tag{2.3.4}$$

on an orthonormal vector tetrad which contains u^α. The projections form a *complex*, trace-free and symmetrical 3×3 matrix‡ (C_{ab}). The linearly independent eigenvectors and their eigenvalues determine the Petrov type. Because of (2.3.3), (C_{ab}) is *real* for IRF solutions and hence can be diagonalised by means of a rotation of the three space-like vectors of the tetrad. Therefore (C_{ab}) possesses three linearly independent eigenvectors; and accordingly *IRF solutions are of Petrov type I, D or* 0.

From (2.2.11d), (2.3.3) and (2.3.4) we find

$$C_{\alpha\beta} = -\tfrac{1}{2}\theta_{\alpha\beta} \tag{2.3.5}$$

Thus the eigenvalues of the viscous stress tensor $\theta^{\alpha\beta}$ *determine the Petrov type of the corresponding IRF solution uniquely as follows*§

Three distinct eigenvalues ⇔ *Petrov type I*

Two distinct eigenvalues ⇔ *Petrov type D*

$$\theta^{\alpha\beta} = 0 \Leftrightarrow \textit{Petrov type } 0 \qquad (\textit{conformally flat}) \tag{2.3.6}$$

† The duality operation is indicated by an asterisk. $[\lambda \ldots \sigma]$ denotes total antisymmetry in all indices $\lambda \ldots \sigma$.

‡ Range of indices: $a, b, \ldots = 1, 2, 3$. Brackets denote a matrix.

§ ⇔ characterises the one-to-one correspondence.

2.4. *Symmetry Properties*

Evidently after (2.1.6), an IRF solution is static and admits at least the following *motion of the metric*†

$$\underset{u}{\mathcal{L}}\, g_{\alpha\beta} = 0 \tag{2.4.1}$$

With regard to the invariances of composed metrical quantities, *an IRF solution admits the three curvature collineations*‡ (Katzin *et al.*, 1970):

$$\underset{\xi_{(a)}}{\mathcal{L}}\, R^{\alpha}_{\beta\gamma\delta} = 0 \tag{2.4.2}$$

wherein

$$\xi^{\alpha}_{(1)} := u^{\alpha}, \qquad \xi^{\alpha}_{(2)} := Su^{\alpha}, \qquad \xi^{\alpha}_{(3)} := S^2 u^{\alpha} \tag{2.4.5}$$

with S given by

$$S_{,\alpha} := u_{\alpha} \tag{2.4.6}$$

They form a three-parameter invariance group, which is characterised by the structure relations

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3 \tag{2.4.7}$$

with the generators

$$X_{(a)} := \xi^{\alpha}_{(a)} \frac{\partial}{\partial x^{\alpha}} \tag{2.4.8}$$

2.5. *Line Element*

From (2.1.6) we easily obtain: *A space-time admits an IRF if and only if its line elements can be put into the (special static) form*‡

$$ds^2 = -g_{ab}(x^c)\, dx^a\, dx^b + dt^2 \tag{2.5.1}$$

choosing $u^{\alpha} = \delta_4{}^{\alpha}$. Every metric of the type (2.5.1) may be taken for an IRF solution. In this case it belongs to an energy momentum tensor, which underlies strongly restricting physical conditions resulting from (2.2.11a–e). We discuss this in the sections that follow.

3. *Vacuum IRF Solutions*

With $T^{\alpha\beta} = 0$, (2.2.11c) and (2.2.11d) lead to

$$R + 2\Lambda = 0 \tag{3.1}$$

respectively

$$R + 6\Lambda = 0 \tag{3.2}$$

which implies

$$R = 0, \qquad \Lambda = 0 \tag{3.3}$$

Consequently, for $\Lambda \neq 0$ only interior IRF solutions are possible.

† $\underset{u}{\mathcal{L}}$ denotes the Lie derivative with respect to u^{α}.

‡ Range of indices: $a, b, \ldots = 1,2,3$.

In the case of vacuum IRF solutions we have from (3.3) and the field equations (2.2.1)

$$R_{\alpha\beta}=0 \tag{3.4}$$

while (2.3.5) leads to

$$C_{\alpha\beta}=0 \tag{3.5}$$

Hence the Weyl tensor vanishes identically. Herewith and after (3.3) and (3.4) it follows from (2.2.4), that the Riemann tensor vanishes too.

$$R_{\alpha\beta\gamma\delta}=0 \tag{3.6}$$

On the other hand, flat space-time certainly is an IRF solution [compare (2.5.1)]. We therefore conclude, that *the unique vacuum IRF solution is the Minkowski space-time.*

4. *Interior IRF Solutions*

4.1. *Restrictions for the Variables of State*

If an interior solution of Einstein's equation (2.2.1) admits an IRF, the corresponding matter possesses no heat flow relative to this IRF, compare (2.2.11b). Furthermore in consequence of (2.2.11c) and (2.2.11d) density ρ and pressure p must satisfy the *very restrictive condition.*

$$\rho+3p=-2\Lambda=\text{const.} \tag{4.1.1}$$

Beyond this, (2.2.11a) yields with (2.2.11c) and (2.2.11d)

$$\rho_{,\epsilon}u^{\epsilon}=0, \qquad p_{,\epsilon}u^{\epsilon}=0 \tag{4.1.2a–b}$$

This means that ρ and p, as measured by an observer in an IRF, are constant with respect to time.

4.2. *Vanishing Cosmological Constant*

Equation (4.1.1) has the following meaning: interior IRF solutions with vanishing cosmological constant belong to matter with the *high negative pressure* $p=-\frac{1}{3}\rho$. For example, matter of density 1 g/cm^3 must have the physically senseless pressure minus 10^{15} atmospheres. It is reasonable, therefore, to conclude that for $\Lambda=0$ only vacuum IRF solutions are possible. With respect to the result of Section 3 this means, that *the only physically meaningfull IRF solution with $\Lambda=0$ is the flat space-time.*

4.3. *Non-Vanishing Cosmological Constant*

If Λ does not vanish (according to Section 3) only interior IRF solutions are possible. In view of this, we discuss three energy momentum tensors that are often used.

Perfect fluid. With respect to the streamlines $\hat{u}^{\alpha}$, the energy momentum tensor of a perfect fluid can be written

$$T^{\alpha\beta}=(\hat{\rho}+\hat{p})\,\hat{u}^{\alpha}\hat{u}^{\beta}-\hat{p}g^{\alpha\beta} \tag{4.3.1}$$

If the corresponding solution of (2.2.1) admits an IRFu^α, equation (2.2.6) takes the form

$$T_{\alpha\beta} = (\hat{\rho} + \hat{p})\,\hat{u}_\alpha \hat{u}_\beta - \hat{p} g_{\alpha\beta} = \frac{R}{3} u_\alpha u_\beta + \left(\frac{R}{6} + \Lambda\right) g_{\alpha\beta} - 2E_{\alpha\beta} \quad (4.3.2)$$

Contraction with u^β gives

$$\left(\frac{R}{2} + \Lambda + \hat{p}\right) u_\alpha = (\hat{\rho} + \hat{p})\,(\hat{u}_\epsilon u^\epsilon)\,\hat{u}_\alpha \quad (4.3.3)$$

This implies ($\hat{p} \neq -\hat{s}$ ensures the existence of streamlines)

$$\hat{u}^\alpha = u^\alpha, \qquad \hat{\rho} = \rho, \qquad \hat{p} = p \quad (4.3.4a–c)$$

In IRF solutions which belong to a perfect fluid, the IRF is represented by the streamlines. This means (in contrast to the situation in the Minkowski space-time), that *only one IRF is possible.*

According to (4.3.1) and (4.3.4a) $\theta^{\alpha\beta}$ vanishes, thus with regard to the Petrov classification we have: *an interior IRF solution belongs to a perfect fluid if and only if it is conformally flat.*

We perform an explicit construction of this IRF solution. It is well known (Stephani, 1967b), that conformally flat solutions of Einstein's equation with a perfect fluid as a source are of the embedding class 1. The special solutions of this class with vanishing expansion of the streamlines are of the type of the interior Schwarzschild solution (Stephani, 1967a). In the coordinate system which is determined by streamlines, $\hat{u}^\alpha \sim \delta_4{}^\alpha$, their line element can be written in the form:

$$ds^2 = -\frac{dr^2}{1 - (r^2/R_0{}^2)} - r^2(\sin^2\vartheta\, d\phi^2 + d\vartheta^2) + (D(r, \vartheta, \phi, t)\, dt)^2 \quad (4.3.5)$$

To ensure the existence of an IRF [compare 4.3.4a–c) and (2.5.1)], we have finally to specialise to $D = \text{const.}$: *the unique IRF solution belonging to a perfect fluid is the (static) Einstein universe.*

Electromagnetic field. The trace of the energy-momentum tensor of the electromagnetic field

$$T_{\alpha\beta} = \tfrac{1}{4} F_{\sigma\tau} F^{\sigma\tau} g_{\alpha\beta} - F_{\alpha\sigma} F_\beta{}^\sigma \quad (4.3.6)$$

vanishes, thus leading to

$$\rho - 3p = 0 \quad (4.3.7)$$

(equation of state). If the corresponding solution of the Einstein equation (2.2.1) admits an IRF, we get from (4.1.1)

$$\rho = -\Lambda = \text{const.}, \qquad p = \text{const.} \quad (4.3.8a–b)$$

and because of $\rho > 0$ *the cosmological constant does not vanish*

$$\Lambda < 0 \quad (4.3.9)$$

Furthermore, we project $T^{\alpha\beta}$ on an orthonormal vector tetrad which contains the IRF vector u^α as the time-like vector. The resulting matrix of the projections can be brought into the following form by rotation of the three space-like vectors (Petrov, 1964)†

$$(T_{\alpha\beta}) = \begin{bmatrix} \frac{a^2+b^2}{2} & 0 & 0 & \pm ab \\ 0 & \frac{-a^2+b^2}{2} & 0 & 0 \\ 0 & 0 & \frac{a^2-b^2}{2} & 0 \\ \pm ab & 0 & 0 & \frac{a^2+b^2}{2} \end{bmatrix} \qquad (4.3.10)$$

According to (2.2.11b) either a or b must be equal to zero. Therefore, putting e.g. $b = 0$, the trace-free 3×3 matrix of the projections of the corresponding viscous stress tensor is of the form†

$$(\theta_{ab}) = \begin{bmatrix} \frac{a^2}{3} & 0 & 0 \\ 0 & -\frac{2}{3}a^2 & 0 \\ 0 & 0 & \frac{a^2}{3} \end{bmatrix} \qquad (4.3.11)$$

Hence, if an IRF solution exists, according to (2.3.6) *the Weyl tensor is of Petrov type D.*

Radiation. The energy-momentum tensor of all sorts of pure radiation is of the type

$$T^{\alpha\beta} = aj^\alpha j^\beta, \qquad j^\epsilon j_\epsilon = 0, \qquad a \neq 0 \qquad (4.3.8)$$

In this case we obtain by contracting (2.2.6) with u_β the following relation

$$(aj^\epsilon u_\epsilon) j^\alpha = \left(\frac{R}{2} + \Lambda\right) u^\alpha, \qquad (j^\epsilon u_\epsilon \neq 0) \qquad (4.3.9)$$

Because this represents a contradiction we conclude, that *pure radiation can not be a source of an IRF solution.*

5. *Conclusions*

The main conclusions of this work are as follows:

(i) $\Lambda = 0$: The only physically admissible IRF solution is the Minkowski space-time. This is also for arbitrary Λ the only mathematically possible vacuum IRF solution.

† Range of indices: $a, b, \ldots = 1, 2, 3$. Brackets denote a matrix.

$\Lambda \neq 0$: Only interior IRF solutions are possible. They belong to a matter distribution which satisfies the very restrictive conditions (4.1.1) and (4.1.2).

(ii) In the case of the perfect fluid, the unique interior IRF solution is the (static) Einstein universe.

(iii) The Petrov type of all IRF solutions is uniquely determined by the corresponding (trace-free) viscous stress tensor according to (2.3.6). Only the types I, D and 0 are possible. The IRF solution belonging to a perfect fluid is conformally flat [compare (ii)], those referring to an electromagnetic field are of type D. For energy-momentum tensors of pure radiation, no IRF solutions are possible.

Acknowledgements

The author wishes to thank Professor Dr. H. Dehnen and Dipl.-Phys. W. Graf for critically reading the manuscript.

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